Grade 7/8 Math Circles

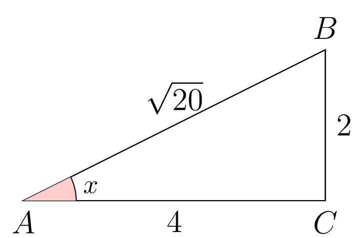
February 12-15, 2024

Trigonometric Ratios - Problem Set

1. Use the Pythagorean Theorem to find the missing side length of a right-angled triangle with a hypotenuse of 23.3 and another side length of 10.5.

> *Solution*: The Pythagorean Theorem states that $c^2 = a^2 + b^2$, where c is the hypotenuse, and a & b are the other two side lengths of a right-angled triangle. We are told that the length of the hypotenuse is 23.3, so we set $c = 23.3$. Next, we have the length of another side is 10.5. Let's call it a; therefore, $a = 10.5$. This leaves us to find b using the Pythagorean Theorem.
>
> $$c^2 = a^2 + b^2$$
> $$(23.3)^2 = (10.5)^2 + b^2$$
> $$(23.3)^2 - (10.5)^2 = b^2$$
> $$b^2 = 432.6 \implies b = \sqrt{432.6} = 20.8$$

2. Consider the triangle $\triangle ABC$ below:



   (a) Verify that $\triangle ABC$ is a right-angled triangle by showing that the Pythagorean Theorem holds, then determine the sine, cosine and tangent ratios of angle x.

   (b) What are three different (but similar) ways we can solve for the value of x?

> *Solution*:
>
> (a) $\triangle ABC$ is a right-angled triangle if the Pythagorean Theorem holds. Let's check by

solving for the hypotenuse ourselves:

$$c^2 = a^2 + b^2$$

$$c^2 = (4)^2 + (2)^2 = 16 + 4 = 20$$

Taking the square root of both sides, we have the hypotenuse $c = \sqrt{20}$. Since the Pythagorean Theorem gives us the same result as in the diagram, $\triangle ABC$ is a right-angled triangle.

The adjacent side has length 4, the opposite side has length 2, and the hypotenuse has length $\sqrt{20}$. So

$$\sin(x) = \frac{O}{A} = \frac{2}{\sqrt{20}}$$

$$\cos(x) = \frac{A}{H} = \frac{4}{\sqrt{20}}$$

$$\tan(x) = \frac{O}{A} = \frac{2}{4} = \frac{1}{2}$$

(b) We can solve for x by using any of the ratios found above. Since $\sin(x) = \frac{2}{\sqrt{20}}$, we can apply the arcsine function to both sides. This tells us $x = \arcsin(\frac{2}{\sqrt{20}}) \simeq 26.6°$. Similarly, we know $\cos(x) = \frac{4}{\sqrt{20}} \implies x = \arccos(\frac{4}{\sqrt{20}}) \simeq 26.6°$. Lastly, we found $\tan(x) = \frac{1}{2} \implies x = \arctan(\frac{1}{2}) \simeq 26.6°$. All three methods give us the same answer of $x = 26.6°$.

3. Create a triangle that has two equal side lengths, an angle of 90°, and some other angle A.
   (a) What is $\tan(A)$? Do such triangles always have the same tangent ratio?
   (b) Solve for A. How could you have done this in your head? (*Hint*: the sum of angles in a triangle is 180°).

Solution:

(a) For simplicity, let's choose the two sides to have a length of 1. Then $\tan(A) = \frac{1}{1} = 1$. If we chose the sides to have length 2, we would get $\tan(A) = \frac{2}{2} = 1$. This pattern continues for any positive number we choose, since we are dividing a number by itself. This will always give us a tangent ratio of 1.

(b) We find A by applying the arctangent function on both sides of the equation:

$$\tan(A) = 1$$

$$\cancel{\arctan}(\cancel{\tan}(A)) = \arctan(1)$$

$$\therefore A = \arctan(1) = 45°$$

We could do this in our head by realizing that since we have a right-angled triangle, the two remaining angles that are **not** 90° must add to $180° - 90° = 90°$ total, since the sum of all angles in a triangle is 180°. Further, since both side lengths are the same, the two angles that sum to 90° must be equivalent. This means each angle is $\frac{90°}{2} = 45°$.

4. Maya walks 8m North then 15m East. How much less distance does she walk if she travels along a straight path from her starting to final position?

Solution:
Once Maya starts walking East, she makes a 90° turn in direction from North, so her path forms a right angle. The distance from her starting to end position is the hypotenuse of the triangle formed. To find her net walking distance, we use the Pythagorean Theorem which tells us:

$$c^2 = a^2 + b^2$$

$$c^2 = (8)^2 + (15)^2 = 64 + 225 = 289$$

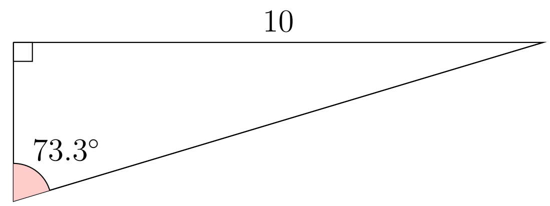
$$\implies c = \sqrt{289} = 17$$

So if Maya were to walk straight to her final position, she would only walk 17m. Her actual path was a total of $8 + 15 = 23$m long. Clearly, this is a longer path to walk since the straight path is only 17m, which is a 6m shorter distance to walk.

5. The area of a right-angled triangle is found by multiplying its base length, b, by its height, h, then dividing this product by 2. The formula is then

$$A = \frac{b \cdot h}{2}$$

Use this to find the area of the following right-angled triangle:



Solution: We have the length of the triangle's base is 10, which is opposite to the given angle of $73.3°$. We are not given the height h, but we know the height is adjacent to the $73.3°$ angle. Since our calculations involve the opposite and adjacent sides about an angle, we will use the tangent ratio, which says

$$\tan(x) = \frac{O}{A}$$

$$\therefore \tan(73.3°) = \frac{10}{h}$$

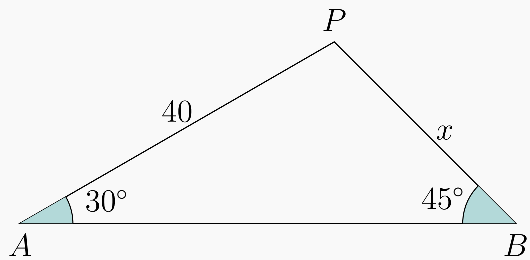
$$\implies h = \frac{10}{\tan(73.3°)} = 3.0$$

Now that we have the height of the triangle, we calculate its area using the formula given.

$$A = \frac{b \cdot h}{2} = \frac{10 \cdot 3}{2} = \frac{30}{2} = 15$$

6. Two pedestrians see a plane in the sky nearby. Pedestrian A sees the plane approaching them from the right at a distance of 40m away at an angle of $30°$ above the ground. Pedestrian B sees the plane flying away from them towards the left at an angle of $45°$ above the ground. How far is pedestrian B from the plane at this instant?

Solution:
We can imagine the situation by drawing a triangle formed by pedestrians A and B and the plane at point P above them.



We recognise that the missing side length x is part of the side-angle pair with 30°, while the 40 and 45° form a complete side-angle pair. Noticing these side-angle pairs should tell us that it is probably helpful to use the Sine Law, which says

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

Plugging in $a = x$, $A = 30°$, $b = 40$ and $B = 45°$, the Sine Law tells us

$$\frac{x}{\sin(30°)} = \frac{40}{\sin(45°)}$$

Multiplying both sides by $\sin(30°)$, we get

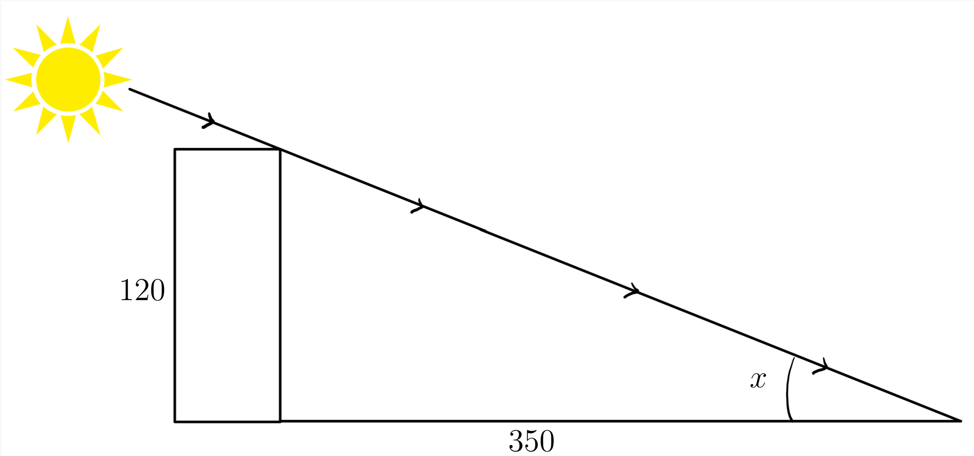
$$\cancel{\sin(30°)} \cdot \frac{x}{\cancel{\sin(30°)}} = \frac{40}{\sin(45°)} \cdot \sin(30°)$$

$$\implies x = \frac{40}{\sin(45°)} \cdot \sin(30°) \simeq 28.3$$

So the plane is 28.3m away from pedestrian B at this instant.

7. The sun emits a ray of light that strikes the top of a 120ft tall building, creating a 350ft long shadow past the building along the ground. What angle does the light make with the ground?

Solution: We can draw a diagram to represent the situation as follows:



We see that the triangle formed from the height of the building, the ground, and the ray of sunlight creates a right-angled triangle. The angle we want, x, has an opposite side length of 120ft and an adjacent side length of 350ft. Since we have sides O and A, we use the TOA in SOH CAH TOA, which tells us

$$\tan(x) = \frac{O}{A} = \frac{120}{350} \simeq 0.34$$

To eliminate 'tan' on the left, we apply the arctangent function to both sides of the equation.

$$\cancel{\arctan}(\cancel{\tan}(x)) = \arctan(0.34)$$

$$\implies x = \arctan(0.34) \simeq 18.8^\circ$$

So the sun makes an angle of 18.8° with the ground.

8. Recall the Cosine Law:

$$c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$$

Its formula is quite similar to the Pythagorean Theorem:

$$c^2 = a^2 + b^2$$

Describe the relationship between the two equations and explain how the Pythagorean Theorem is just a special case of the Cosine Law for positive values of a, b & c. What must angle C be?

Solution:
We can see that the formulas are exactly the same except for the last term in the Cosine Law. To make them the same, we would need the last term to disappear, i.e.

$$-2ab \cdot \cos(C) = 0$$

Since a and b can be any non-zero value, let's solve for what C must be by dividing by $-2ab$ on both sides:

$$\frac{\cancel{-2ab} \cdot \cos(C)}{\cancel{-2ab}} = \frac{0}{-2ab}$$
$$\therefore \cos(C) = 0$$

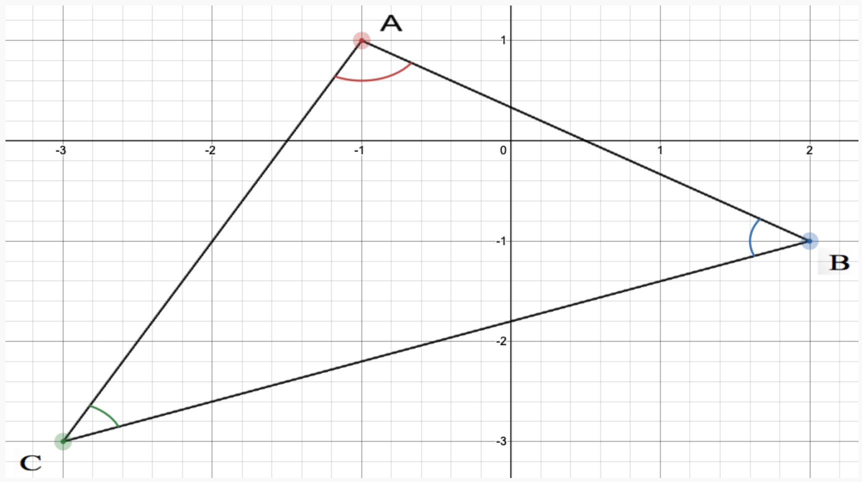
It may be helpful to imagine this result more visually. We see that for the two formulas to be equal, $\cos(C)$ must equal 0. Thinking back to SOH CAH TOA, the cosine of an angle C was defined as $\cos(C) = \frac{A}{H}$, which equals zero only if $A = 0$. However, if $A = 0$, that just means we have a straight line that only moves vertically (zero distance to the side). This is exactly what a right angle is!
If $C = 90°$, the Cosine Law is reduced to just the Pythagorean Theorem since we must be dealing with a right-angled triangle. We should expect this result since we can only use the Pythagorean Theorem when we have a right-angled triangle, whereas the Cosine Law let's us solve any triangle we want, regardless of whether we have a right-angles triangle or not. The Cosine Law is a generalization of the Pythagorean Theorem so we can solve triangles that don't necessarily have any right angles.

9. A triangle is placed on a grid. Its three vertices on the (x, y) plane are point A at the coordinates $(-1, 1)$, B at $(2, -1)$, and C at $(-3, -3)$. Find the total perimeter of the triangle without finding any angles. (*Hint:* the distance between any two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

simply from using the Pythagorean Theorem).

Solution: The triangle formed is shown below.



We can find the length of each side by splitting the triangle into three right-angled triangles and solving for each hypotenuse. This is the same as using the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
Let's first find side length $\overline{AB}$. The coordinates of A and B are $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (2, -1)$, respectively. Using the distance formula in the hint as follows, we can solve for the distance between the coordinates A and B.

$$\overline{AB} = \sqrt{(2-(-1))^2 + (-1-1)^2} = \sqrt{(3)^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13} \simeq 3.61$$

For side $\overline{AC}$, we are given the coordinates A at $(-1, 1)$ and C at $(-3, -3)$. We can find the distance between them by calculating

$$\overline{AC} = \sqrt{(-3-(-1))^2 + (-3-1)^2} = \sqrt{(-2)^2 + (-4)^2} = \sqrt{4+16} = \sqrt{20} \simeq 4.47$$

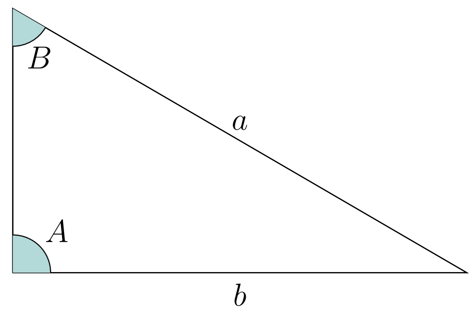
Finally for side $\overline{BC}$, we have points B and C are $(2, -1)$ and $(-3, -3)$, respectively. Plugging these numbers into our distance formula, we get

$$\overline{BC} = \sqrt{(-3-2)^2 + (-3-(-1))^2} = \sqrt{(-5)^2 + (-2)^2} = \sqrt{25+4} = \sqrt{29} \simeq 5.39$$

The perimeter is simply the sum of each side length. So the total perimeter P of $\triangle ABC$ is

$$\overline{AB} + \overline{AC} + \overline{BC} \simeq 3.61 + 4.47 + 5.39 = 13.47$$

10. Consider the triangle below.



(a) What does the Sine Law tell us about this triangle?

(b) Given that $\sin(90°) = 1$, use (a) to prove the SOH in SOH CAH TOA. That is, for a right-angled triangle, the sine of an angle within the triangle is equal to the ratio of the angle's opposite side to the triangle's hypotenuse.

Solution:

(a) We have two side-angle pairs a & A, and b & B. The Sine Law tells us that

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

(b) It appears that A may be a right angle. Let's imagine this is true, implying we have a right-angled triangle with hypotenuse a. Substituting $A = 90°$ into our equation above, we get

$$\frac{\sin(90°)}{a} = \frac{\sin(B)}{b}$$

However, we are told that $\sin(90°) = 1$, so our equation can be simplified to

$$\frac{1}{a} = \frac{\sin(B)}{b}$$

$$b \cdot \frac{1}{a} = \frac{\sin(B)}{\cancel{b}} \cdot \cancel{b}$$

$$\frac{b}{a} = \sin(B)$$

This ratio tells us that the sine of our angle B is equal to the ratio of its opposite side b to the hypotenuse a. But recall that this is exactly what SOH CAH TOA tells us: given a right-angled triangle, the sine of an angle in the triangle is equal to the opposite side length divided by the hypotenuse. Therefore, we have proved that if A

is a right angle, then the sine of another angle in the triangle is simply its opposite side divided by the hypotenuse, as desired.

11. An equilateral triangle is any triangle that has all three sides of equal length. Use the Cosine Law to show that each angle inside any equilateral triangle is $60°$.

Solution:
The Cosine Law states $c^2 = a^2 + b^2 - 2ab \cdot \cos(C)$, where a, b and c are side lengths, and C is the angle in which c is opposite to. If all three sides have the same side length, we can replace both a and b with c in the Cosine Law. Also consider that since all sides are the same, all angles must also be the same. We would then have

$$\cancel{c^2} = \cancel{c^2} + c^2 - 2cc \cdot \cos(C)$$

$$0 = c^2 - 2c^2 \cos(C))$$

If we divide both sides by c^2, we get

$$\frac{0}{c^2} = \frac{\cancel{c^2} - 2\cancel{c^2}\cos(C)}{\cancel{c^2}}$$

$$0 = 1 - 2\cos(C))$$

$$2\cos(C) = 1$$

$$\therefore \cos(C) = \frac{1}{2}$$

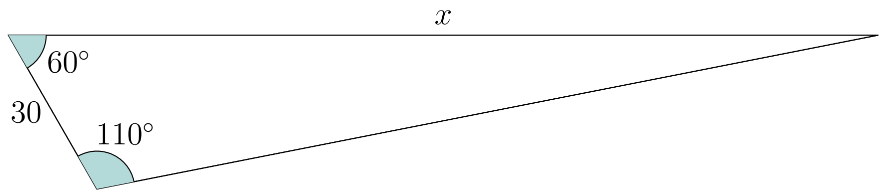
Finally solving for C, we take the arccosine of both sides.

$$\cancel{\arccos(\cos}(C)) = \arccos\left(\frac{1}{2}\right)$$

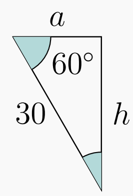
$$C = 60°$$

However, we know all angles in the triangle must be equivalent, therefore $A = B = C = 60°$.

12. * Solve for x using only the Pythagorean Theorem and/or SOH CAH TOA. Verify your answer with either the Sine or Cosine Law. Which method is simpler?



Solution: We can split the triangle into two right-angled triangles.
The entire side length x can be found by adding the horizontal side lengths of our two right-angled triangles, a and b. To start, let's solve the first triangle. Let's call the height of the triangle h.



We see the hypotenuse is 30, and the given angle is 60°. Using SOH CAH TOA, we have

$$\cos(60^\circ) = \frac{A}{H} = \frac{a}{30} \implies a = 30 \cdot \cos(60^\circ) = 15.0$$

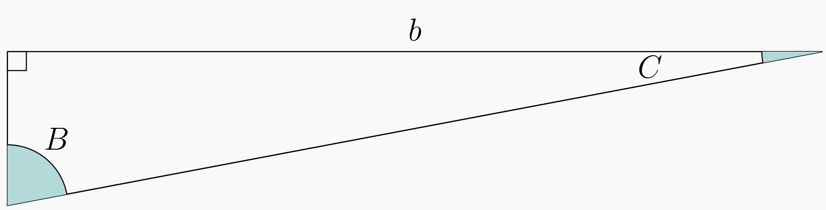
and

$$\sin(60^\circ) = \frac{O}{H} = \frac{h}{30} \implies h = 30 \cdot \sin(60^\circ) = 26.0$$

We can also find the unknown angle by subtracting the two known angles from 180°. If we call this angle A, then

$$A = 180^\circ - 90^\circ - 60^\circ = 30^\circ$$

Let's now focus on our next triangle.



We know its height is 26.0 which is the opposite side to angle C, but we can also find all of the angles forming the triangle. Notice that the entire bottom angle of the original triangle is 110°. We found the portion of this angle inside of the first triangle is $30^\circ = A$, which means the remaining portion of the 110° angle is $110^\circ - 30^\circ = 80^\circ = B$. Since both triangles are right-angled triangles, the final angle C is just $180° - 90° - 80° = 10°$. We can now use either B or C to solve for the side length b with the help of SOH CAH TOA. Let's choose to use the $80°$ angle. Since we are not using the hypotenuse in our calculations, we will use TOA, which tells us

$$\tan(80°) = \frac{O}{A} = \frac{b}{26}$$

$$\implies b = 26 \cdot \tan(80°) \simeq 147.3$$

Finally, we add a and b to find $x = a + b = 15.0 + 147.3 = 162.3$.
An easier approach is to use the Sine Law. The entire triangle has $180°$, so angle $C = 180° - 110° - 60° = 10°$. This is the same as what we had before by splitting the triangle into two right-angled triangles. Now, we can use the Sine Law to find x directly. Inputting our side-angle pairs into the formula, we have

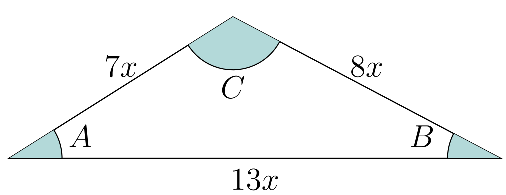
$$\frac{30}{\sin(10°)} = \frac{x}{\sin(110°)}$$

$$\implies x = \sin(110°) \cdot \frac{30}{\sin(10°)}$$

$$\simeq 162.3$$

Exactly the same as before! However, using the Sine Law required less than half of the calculations we needed for the original method. We get the same result, but much quicker.

13. ** The perimeter of the following triangle is 56cm. Solve for each side length and angle. Do the angles change for different values of x?



Solution:
We are told the perimeter of the triangle is 56cm. Adding all of the side lengths together

tells us the perimeter is

$$a + b + c = 7x + 8x + 13x = 28x$$

$$\therefore 56 = 28x \implies x = 2$$

This means that $a = 7x = 7 \cdot 2 = 14$, $b = 8x = 8 \cdot 2 = 16$ and $c = 13x = 13 \cdot 2 = 26$. Now that we've solved for all of the sides, we can use the Cosine Law to find some angles. Let's first find angle A:

$$a^2 = b^2 + c^2 - 2bc \cdot \cos(A)$$

$$(16)^2 = (14)^2 + (26)^2 - 2(14)(26) \cdot \cos(A)$$

$$(16)^2 - (14)^2 - (26)^2 = -2(14)(26) \cdot \cos(A)$$

$$\frac{(16)^2 - (14)^2 - (26)^2}{-2(14)(26)} = \frac{\cancel{-2(14)(26)} \cdot \cos(A)}{\cancel{-2(14)(26)}}$$

$$\therefore \cos(A) = 0.85$$

This is the cosine of A, so let's take the arccosine of both sides to solve for angle A itself:

$$A = \arccos(0.85) = 32.2^\circ$$

Now that we have one angle, we can either repeat the same process for angles B and C, or we can use the Sine Law. For practice, let's try using the Sine Law where $a = 16$, $A = 32.2^\circ$ and $b = 14$. We are trying to solve for B. Using the Sine Law, we have

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

$$\frac{\sin(32.2^\circ)}{16} = \frac{\sin(B)}{14}$$

$$\implies \sin(B) = 14 \cdot \frac{\sin(32.2^\circ)}{16} \simeq 0.47$$

$$\therefore B = \arcsin(0.47) \simeq 27.8^\circ$$

The only thing left to solve is angle C. We can use either the Sine or Cosine Law, or we could simply recognize that the sum of angles in a triangle is $A + B + C = 180^\circ$. Rearranging for C, we find

$$C = 180^\circ - A - B$$

$$= 180° - 32.2° - 27.8°$$

$$= 120.0°$$

Let's notice that the value we found for x above does not actually affect the angles. We say that angles in a triangle are 'invariant under scaling'. If we did not solve for x and plugged our side lengths into the Cosine Law formula, we would have

$$(8x)^2 = (7x)^2 + (13x)^2 - 2(7x)(13x) \cdot \cos(A)$$

$$64\cancel{x^2} = 49\cancel{x^2} + 169\cancel{x^2} - 182\cancel{x^2} \cdot \cos(A)$$

$$64 = 49 + 169 - 182 \cdot \cos(A)$$

We see that every term with x^2 cancels. By doing this, we've shown that as long as $x > 0$, the value of x will not impact our angle values.